

# Choptuik Scaling and The Merger Transition

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**Barak Kol**

*Racah Institute of Physics*

*Hebrew University*

*Jerusalem 91904, Israel*

`barak_kol@phys.huji.ac.il`

**ABSTRACT:** The critical solution in Choptuik scaling is shown to be closely related to the critical solution in the black-string black-hole transition (the merger), through double analytic continuation, and a change of a boundary condition. The interest in studying various space-time dimensions  $D$  for both systems is stressed. Gundlach-Hod-Piran off-critical oscillations, familiar in the Choptuik set-up, are predicted for the merger system and are predicted to disappear above a critical dimension  $D^* = 10$ . The scaling constants,  $\Delta(D)$ ,  $\gamma(D)$ , are shown to combine naturally to a single complex number.

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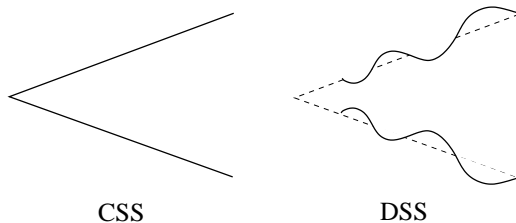
*To my brothers,  
Boaz, Amos, Amir and Yohai.*

## 1. Introduction

This work describes a correspondence between two gravitational systems: Choptuik scaling and the “merger” transition in the black-hole black-string system.

Choptuik scaling [1] describes the famous phenomenon observed (in numerical simulations) at the threshold for black hole production in a spherically symmetric gravitational collapse. At threshold, also known as criticality, the solution approaches an attractor solution as one approaches the space-time point where the black hole is “marginally” formed. The observed independence of initial conditions (as long as one tunes one parameter for criticality) is known as “universality”. The solution has discrete self-similarity, known as “echoing”, with scaling constant denoted by  $e^\Delta$ , and it exhibits a “critical exponent”,  $\gamma$ . See also the review [2] and references therein.

The merger transition originates in the black-string black-hole transition, which occurs whenever extra compact dimensions are present (see [3], the review [4] and references therein). Instead of analyzing the full time-evolution during phase transition it suffices, for purposes of determining the end-state, to consider only stable static solutions, and it turns out to be convenient to consider unstable static solutions as well. For “phase conservation” reasons it was first predicted [3] and recently numerically confirmed [5] that there exists a path of solutions joining the branch of increasingly non-uniform black-strings with the black hole branch. Locally, at the point of minimal horizon radius, or “waist”, a topology



**Figure 1:** An illustration of a continuously self-similar geometry (CSS), which we often call a cone, and a discretely self-similar geometry (DSS).

changing transition occurs, where not only the horizon topology changes, but actually the manifold topology changes as well (at least in the Euclidean, Wick rotated solutions, gotten from the static solutions). This transition was called “merger”, since it can be thought to describe the merger of a large enough black-hole into a black-string.

After reviewing the Choptuik and merger systems in section 2, it is shown in section 3 that the two are closely related, as anticipated in [3] (p.21 bottom of page), and more recently in [6]<sup>1</sup> in connection with double analytic continuation (see [7, 8] for related discussions) of D-branes. In both systems space-time is effectively 2d after accounting for symmetry, and once the Choptuik scalar in  $d$  dimensions is interpreted as arising from a Kaluza-Klein reduction in  $D = d + 1$  dimensions, they are seen to have the same matter content. Moreover, I show that once one performs a *double analytic continuation* the two systems have precisely the same action. It should be noted however, that the two analytic continuations are of a different character: one is trivial in the sense that the fields do not depend on the rotated coordinate, while the other is non-trivial, involving an essential coordinate, one which the fields depend on, and the success of the rotation (reality of the solution) relies on the fields being even in that coordinate.

In order for the solutions to correspond under double analytic continuation, it is not enough that the actions coincide (and therefore the equations of motion) but the boundary conditions (b.c.) must correspond as well. In both cases we are seeking a local solution near a point-like singularity. In Choptuik it is the marginal black hole and in the merger it is the marginally pinched horizon. Locality means that all scales are forgotten near the singularity and thus scale periodicity is a common b.c. Alternatively, the periodic b.c. may be replaced by the attractor mechanism where in both cases it is necessary to fine-tune one parameter – in Choptuik it is the initial imploding wave while in the merger it is a b.c. such as the temperature that parametrizes the curve of solutions. These are boundary conditions along the “scaling direction”, but we still need b.c. along the “tangential” direction. There one actually finds two kinds of b.c., and in this respect the two systems differ.

Self-similar solutions, such as the critical solutions discussed here, can be either Continuously Self-Similar (CSS) or Discretely Self-Similar (DSS). CSS solutions return to themselves after rescaling by any constant and a cone is a good mental picture for them, while DSS solutions are invariant only by a rescaling by a specific rescaling factor (and its power)

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<sup>1</sup>I thank Nissan Itzhaki for introducing me to the observation made in that paper.

	Merger	Choptuik Critical Collapse
(Essential) Signature	Euclidean	Lorentzian
Boundary condition	Reflection	Axis regularity
Self Similarity of critical solution	Continuous <sup>2</sup>	Discrete

**Table 1:** A summary of the differences between the critical merger solution and the critical Choptuik solution.

and can be represented by a wiggly cone with logarithmically periodic wiggles (see figure 1). In subsection 3.4 we discuss CSS solutions and especially the evidence [23] for the double-cone (which is CSS) being the critical merger solution. That leads to an interpretation of the complex exponents that appear in the perturbations of the double cone [3] as critical exponents of the merger system. The real part is predicted to be related to the critical exponent  $\gamma$ , which sets the dimensions of the off-criticality parameter  $(p - p^*)$ , while the imaginary part is predicted to be related to the critical exponent  $\Delta$ . In critical collapse, on the other hand, the critical solution is DSS. While there  $\Delta$  manifests itself as the log-periodicity of both the critical solutions and the Gundlach-Hod-Piran (GHP) oscillations of off-critical quantities, for the merger we predict the latter without the former. Altogether, the scaling constants  $\gamma$ ,  $\Delta$  are shown to combine naturally into a single complex number related to the perturbative exponents, the precise relation being (3.16), which generalizes the well-known connection of  $\gamma$  with the linearized analysis (2.11).

The differences between the merger system and the Choptuik critical collapse are summarized in table 1. Some implications are discussed in section 4. Briefly, they are

- A prediction of GHP oscillations in the merger system.
- The cone provides a prediction of the critical exponents  $\Delta$ ,  $\gamma$  and a critical dimension  $D^* = 10$  for the merger (3.19-3.21).

This prediction has some analogy with [9, 10] which analytically estimate the critical exponents of Choptuik scaling by analyzing perturbations around the CSS Roberts solution [11] in 4 and higher dimensions, respectively.

- Perhaps there are similarities between the solutions and scaling constants of the two systems as they differ only by a change of b.c. and therefore perhaps some results would carry over from the merger to the standard Choptuik system.

**Distant outlook.** Finally, I would like to discuss some general but non-rigorous lessons

- Choptuik scaling is well-known to be very similar to conformal field theories, as it exhibits scale invariance and critical exponents. Equipped with the modern perspectives of holography, and the duality between 2d gravity and matrix models, I find it suggestive to predict that quantum gravity near the singularity (for both the spherical collapse and the merger) is described by some (yet unknown) large  $N$  *conformal matrix model*.

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<sup>2</sup>Direct evidence is lacking so far.

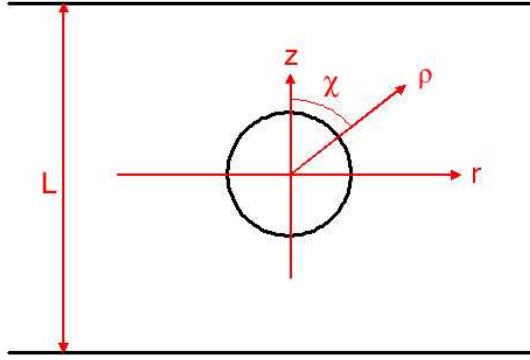
- The time evolution during the Gregory-Laflamme decay inevitably leads to a pinching singularity with high energy effects, irrespective of the (low energy) initial conditions. It is very probable that this singularity would exhibit scaling as well. In that sense it is a manifestation of *self-organized criticality*. It would be interesting to understand the critical phenomena near this singularity which distinguishes itself by involving three essential coordinates –  $(r, z, t)$ , rather than two (see [12] for a simulated time evolution).

**Note added** (version 3). The perspective gained by developments including [21, 22, 23] proves the main results to be

- The relation between the merger and Choptuik scaling through double analytic continuation (section 3).
- The prediction of GHP oscillations for the merger system in subsection 3.4 with critical exponents given by (3.19-3.21).

Moreover, following the same developments some of the questions raised in the original “implications” section were answered as we proceed to describe. The body of the paper, sections 2,3 are mostly unchanged, while sections 1,4 were changed accordingly.

- The suggestion in version 1 (v1) to study Choptuik’s critical collapse in various dimensions was taken up successfully in [21] (see section 4 for a discussion).
- The idea in v1 to numerically study the analytical continuation of either merger or spherical collapse was conditioned on the stability of the analytically continued solutions. I came to believe that these continuations are unstable and so would be any numerical implementation of them [22]. Instabilities are typical in the analytic continuation of actions – the essential argument can be seen in the harmonic oscillator Lagrangian  $L \sim \dot{x}^2 - x^2$ . If one performs an analytic continuation  $t \rightarrow i t$  the kinetic term is inverted, and after a multiplication of  $L$  by an overall minus sign we find that the potential term is inverted leading to instability.
- [23] showed that the double cone is an attractor at codimension 1 for a certain class of admissible perturbations and that was interpreted as significant evidence that it is the critical merger solution. Therefore the critical solution is probably CSS, rather than DSS as originally predicted in v1. The cause for the error was that originally, the off-critical oscillations of the merger were taken to imply DSS, just like in the critical collapse the two measure the same log-period  $\Delta$ . However, it turns out that the implication is only in the converse direction, namely DSS implies GHP oscillations, and the latter can exist even without the former. In particular, the perturbations are not interpreted here as tachyons anymore. Subsection 3.4 was changed accordingly.



**Figure 2:** Definition of coordinates for the merger system. For backgrounds with a single compact dimension the essential geometry is 2d and Euclidean after suppressing the time  $t$  and angular coordinates in the extended dimensions. The cylindrical coordinates  $(r, z)$  are defined such that  $z \sim z + L$  is the coordinate along the compact dimension and  $r$  is the radial coordinate in the extended spatial directions. Locally at the “pinching singularity” we define another set of local coordinates  $(\rho, \chi)$  (defined only for  $\rho \leq L/2$ ), which are radial coordinates in the 2d plane with origin at the singularity. We shall sometimes call  $\rho$  a “scaling coordinate” and  $\chi$  “tangential”.

## 2. Review

We start by reviewing the two concepts to be linked in this work.

### 2.1 The merger transition

As reviewed in the introduction the merger transition originates in the black-string black-hole transition, which occurs whenever extra compact dimensions are present (see [3], the review [4] and references therein). The merger transition is the local topology change at the “waist” as one moves from an unstable black-string to an unstable black-hole.

**The system.** One considers black objects in a flat  $D$ -dimensional space-time background  $\mathbb{R}^{D-2,1} \times \mathbf{S}^1$ , where the compact coordinate is denoted by  $z$  and its length is  $L : z \sim z + L$  (see figure 2). The matter content is pure gravity and the action is the standard Einstein-Hilbert action  $S_D = 1/(16\pi G) \int R \sqrt{g} d^D x$ .

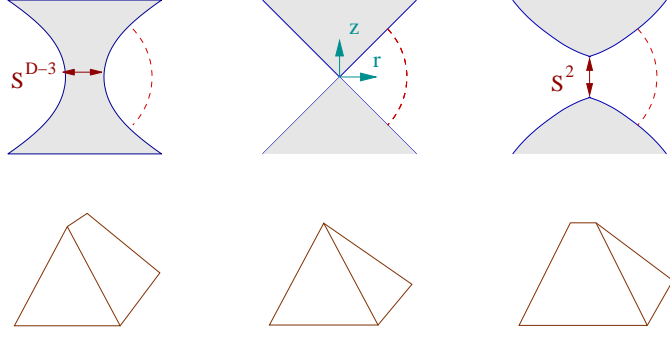
The black objects are spherical and static, namely the isometry is  $SO(D-2)_\Omega \times U(1)_t$  (“static” means also time reversal symmetry). The most general metric consistent with this symmetry is

$$ds^2 = -e^{2A} dt^2 + ds_{(r,z)}^2 + e^{2C} d\Omega_{D-3}^2, \quad (2.1)$$

where all fields are defined on the Euclidean  $(r, z)$  plane,  $ds_{(r,z)}^2$  is an arbitrary metric on the plane and since the metric is static an analytic continuation  $t \rightarrow it$  is trivial and we may freely switch between the Euclidean and Lorentzian signatures.

Altogether the problem is defined in the Euclidean  $(r, z)$  plane and the field content is a 2d metric and two scalars  $A, C$ . That means that we can write down a 2d action for these fields without loosing any of the equations of motion. The action is

$$S = \frac{\beta L}{4G_D} \int dV_2 e^{A+2C}.$$



**Figure 3:** The merger transition. A black string (left) turns into a black hole (right) as the waist pinches. Shaded regions are inside the horizon and the dashed line is a boundary far away. The singular configuration is a cone over  $\mathbf{S}^2 \times \mathbf{S}^{D-3}$  – the double-cone.

$$\cdot \left[ R_2 + (D-3)(D-4) e^{-2C} + (D-3)(D-4) (\partial C)^2 + 2(D-3) (\partial A)(\partial C) \right] \quad (2.2)$$

where  $R_2$  is the 2d Ricci scalar and  $dV_2 := \sqrt{g_2} dr dz$  is the volume element. The total number of fields is 5. Two fields may be eliminated by a choice of coordinates in the plane which leaves us with three fields.

**The double-cone.** Intuitively the transition from black string to black hole involves a region where the horizon becomes thinner and thinner as a parameter is changed until it pinches and the horizon topology changes. This region is called “the waist” and this process is described in the upper row of figure 3 using the  $(r, z)$  coordinates defined in figure 2. It is important to remember that all metrics under consideration are static and that they change as we change an external parameter, not time.

A topological analysis [3] indicates that *the local topology of (Euclidean) spacetime is changing*, not only the horizon topology. Moreover, *the topology change can be modeled by the “pyramid” familiar from the conifold transition* (see the lower row of figure 3). By the nature of topology, in order to change it there must be at least one singular solution along the way (with at least one singular point). The simplest possibility, which is also realized in the conifold is to assume that *the singular topology is the cone over  $\mathbf{S}^{D-3} \times \mathbf{S}^2$* , which we term *the “double-cone”*.

It is easy to write down a Ricci flat metric for the singular solution, which is moreover continuously self-similar (CSS).<sup>3</sup> The metric is

$$ds^2 = d\rho^2 + \frac{\rho^2}{D-2} \left[ d\Omega_{\mathbf{S}^2}^2 + (D-4) d\Omega_{\mathbf{S}^{D-3}}^2 \right] , \quad (2.3)$$

where the  $\rho$  coordinate measures the distance from the tip of the cone, which is the only singular point, and the constant pre-factors are essential for Ricci-flatness.

It turns out that the double-cones may have oscillating perturbations and that their existence surprisingly depends on a critical dimension  $D^* = 10$  [3]. The relevant mode is

<sup>3</sup>We shall freely interchange the terms “cone” and “CSS”.

a function  $\epsilon(\rho)$  which inflates slightly one of the spheres while shrinking the other. The ansatz for the perturbation is

$$ds^2 = d\rho^2 + \frac{\rho^2}{D-2} \left( e^{2\epsilon/2} d\Omega_{\mathbf{S}^2}^2 + (D-4) e^{-2\epsilon/(D-3)} d\Omega_{\mathbf{S}^{D-3}}^2 \right). \quad (2.4)$$

*A priori* one could start with two separate scale functions, one for each sphere, but a constraint relates them as above.

Considering the zero mode for  $\epsilon$ , namely linearized deformations around the double-cone, one finds from the equations of motion for the ansatz (2.4)

$$\begin{aligned} \epsilon &= \rho^{s_{\pm}} \\ s_{\pm} &= \frac{D-2}{2} \left( -1 \pm i \sqrt{\frac{8}{D-2} - 1} \right). \end{aligned} \quad (2.5)$$

The imaginary part,  $\Im(s)$ , causes the oscillations. For  $D \geq D^* := 10$  we see that  $s_{\pm}$  become purely real, namely  $D^* = 10$  is a *critical dimension*.

## 2.2 Choptuik scaling

Consider the threshold for black hole production. It is a co-dimension 1 surface (or “wall”) in the space of initial conditions of any gravitational theory. “Choptuik scaling” is the term for the critical phenomena physics at this threshold. Here we shall review the basic system where the famous discoveries of Choptuik were made through computerized simulations of spherical collapse [1] and describe its salient features. More information, and a survey of other systems can be found in the excellent review [2].

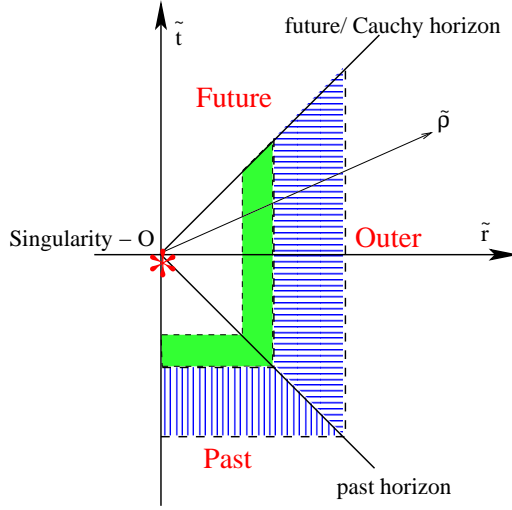
**The system.** One considers an implosion of a spherical shell. There is a small price to pay for the high degree of symmetry - the shell cannot be made of gravitational waves (which do not possess an S-wave due to their spin 2). A simple choice for the matter content is a single scalar field,  $\Phi$ . Thus the action is taken to be

$$S_{\text{Choptuik}} = \frac{1}{16\pi G_N} \int \sqrt{-g} d^d x \left( R + \frac{1}{2} (\partial\Phi)^2 \right) \quad (2.6)$$

Spherical symmetry means that the essential coordinates (upon which the fields depend),  $(\tilde{r}, \tilde{t})$  parametrize a 2d Lorentzian plane (see figure 4). We use tilded coordinates for the Choptuik solutions to distinguish them from the untilded coordinates for the merger.

Then one considers a family of initial conditions parametrized by some parameter  $p$ . For instance, one could take a family of Gaussian-profiled scalar waves, or any other profile, with  $p$  being proportional to the initial amplitude. For small enough  $p$  the linear approximation is valid, and by superposition the waves go through the origin and “reflect” back to infinity. For large enough  $p$  a black hole forms. Thus, for any such family the threshold of black hole formation defines a critical value of the parameter which is denoted by  $p_*$ . Naturally, the value of  $p_*$  depends on the chosen family.





**Figure 4:** Definition of coordinates for spherical collapse system and Choptuik scaling (based on [13, 14]). The essential coordinates  $(\tilde{r}, \tilde{t})$  parametrize a 2d Lorentzian plane after suppressing the angular coordinates. The scaling direction is along lines of fixed  $\tilde{t}/\tilde{r}$  and may be parametrized by  $\tilde{\rho}^2 := \tilde{r}^2 - \tilde{t}^2$ .  $\tilde{z}$  parametrizes an additional dimension, the dimensional uplift of the scalar field  $\Phi$ . The domain is made out of three patches: the past patch, bounded by the  $\tilde{r} = 0$  axis and the past horizon, the outer patch bounded by the past and future horizons and the future patch bounded by the future (Cauchy) horizon and the axis. The critical solutions is periodic on smaller and smaller scales as the singularity is approached. One period is denoted by the line-filled (blue) region and a second one is denoted by a shaded (green) regions. The pattern continues towards the singularity.

**Main results.** It was found that this system displays *universality*, namely, some properties are independent of the chosen family of initial conditions. There are two main universal quantities

- The critical exponent  $\gamma$ .
- The log-periodicity  $\Delta$ .

*The critical exponent.* Consider the black-hole mass as a function of  $p$ , namely  $M_{BH} = M_{BH}(p)$ . For  $p < p_*$   $M_{BH} = 0$  while for  $p > p_*$   $M_{BH} > 0$  and thus  $p = p_*$  is a non-analytic point of this function. It turns out that for  $p \gtrsim p_*$

$$M_{BH} \simeq (p - p_*)^\gamma . \quad (2.7)$$

where  $\gamma$  is a universal critical exponent. In 4d [13]

$$\gamma \simeq 0.374 . \quad (2.8)$$

*Echoing and log-periodicity.* For  $p = p_*$  there is a special point in space-time where the black-hole is “just almost” being created. Clearly it is at  $\tilde{r} = 0$  and we might as well shift

$\tilde{t}$  so that it has  $\tilde{t} = 0$ . Let us denote this point by  $O$ . As the simulation approaches  $O$  the solution starts repeating itself on shorter scales and on shorter times. More precisely the solution approaches the “critical solution”,  $Z_*$ , independently of initial conditions, and  $Z_*$  is discretely self-similar (DSS) with  $O$  being its fixed-point.

In pictures, DSS means that the space looks like an inhomogeneous cone - a cone that was deformed in a periodic manner, see figure 1. In formulae, DSS means that there is a transformation on space-time  $x \rightarrow x' = f(x)$  such that the solution is invariant up to rescaling

$$\begin{aligned} g'_{\mu\nu}(x) &= e^{-2\Delta} g_{\mu\nu}(x) \\ \Phi(x') &= \Phi(x) + \kappa_\phi, \end{aligned} \tag{2.9}$$

where  $g'$  is the induced metric  $g_{\mu\nu} \rightarrow g'_{\mu\nu}(x) = f^*(g_{\mu\nu})(x')$ . The scalar field allows for a shift constant  $\kappa_\phi$  consistent with DSS, but in the Choptuik critical solution,  $\kappa_\phi = 0$  “for unknown reasons” [2]. In standard coordinates  $f$  is conveniently given by  $f(\tilde{r}, \tilde{t}) = (e^{-\Delta} \tilde{r}, e^{-\Delta} \tilde{t})$ , or equivalently  $\tilde{\rho} \rightarrow e^{-\Delta} \tilde{\rho}$ . The log-period  $\Delta$  was numerically measured in 4d to be

$$\Delta(d=4) \simeq 3.45. \tag{2.10}$$

See table 2 for a collection of these and other measurements.

Continuous Self Similarity (CSS) would mean for a geometry to have a transformation satisfying (2.9) for all  $\Delta$ , and that  $\kappa_\phi = \text{const } \Delta$ .

Universality is a consequence of  $Z_*$  being an attractor on the co-dimension 1 surface in phase space. Actually, for DSS the attractor is a “limit-cycle”.

Discovering “echoing” required special determination and high-quality numerics, since it was necessary to follow the solution for several periods, each period lasting a factor of  $e^\Delta \simeq 30$ .

*Analogy with second order phase transitions.* The two phenomena of critical exponents and scale invariance are the hallmark properties of second order phase transitions in field theory (for instance, the Curie transition in ferromagnetic materials, or the liquid-gas critical point). There the critical point is described by a Conformal Field Theory that looks the same on all scales. If one deviates from the conformal point the power law behavior of correlation functions is replaced by a finite correlation length. Critical exponents appear which are related to the dimensions of certain operators in the conformal point. Moreover, reaching the CFT requires tuning some control parameters, much as the Choptuik solution is gotten after fine-tuning the initial conditions. Despite this strong analogy with CFT, a CFT is not known to appear in Choptuik scaling. Given the modern perspective of Holography via the AdS/CFT correspondence and the duality of 2d gravity and matrix models, it is natural to predict that the critical solution is dual to a large  $N$  conformal matrix model.

**Other results.** To date there has been little analytic understanding of  $\gamma$ ,  $\Delta$ . The main information added is the relation

$$\gamma = 1/\lambda_0 \tag{2.11}$$

where  $\lambda_0$  is the unique negative eigenvalue of the critical solution. The reason that there is such a unique eigenvalue is that  $Z_*$  being an attractor throughout the co-dim 1 black-hole-threshold “wall” is stable to all perturbations within the wall (stability means positive eigenvalues), but is unstable against a deviation outside of it.

The point  $O$  is a naked singularity.  $O$  is singular as an immediate consequence of the scaling symmetry - as we get closer to  $O$  the curvature scales up and is unbounded in its neighborhood. Moreover, there is no horizon yet as we are at the threshold of black hole formation<sup>4</sup>. This teaches us that naked singularities are generic at co-dim 1 in phase space, and the celebrated Cosmic Censorship conjecture must be amended to read “no naked singularities will form for “reasonable” *and generic* initial conditions”.

The behavior of  $M_{BH}$  around  $p_*$  gets a sub-leading periodic correction in DSS

$$\log(M_{BH}) = \gamma \log(p - p_*) + c + f_{GHP}(\gamma \log(p - p_*) + c) , \quad (2.12)$$

where  $f_{GHP}$  is a universal function (our notation, GHP, stands for Gundlach-Hod-Piran [13, 15]) with period  $\Delta$ , while  $c$  depends on initial conditions.

Another peculiar phenomenon is that when one inspects the metric alone, without the scalar field, one finds that the log-frequency doubles, namely the log-period for the metric is  $\Delta/2$ . For the scalar field, on the other hand, only odd frequencies are present.

### 3. The correspondence

In this section the central claim is stated:

**Claim:** The critical merger solution in  $D$  dimensions corresponds after a double analytic continuation to a variant of the critical Choptuik solution in  $d = D - 1$  dimensions but with different b.c.: time reversal symmetry replaces axis regularity.

In order to prove this claim we first demonstrate that the actions are the same up to a double analytic continuation, and therefore the equations of motions are identical. Then we analyze the boundary conditions to show that the solutions are identical as well.

#### 3.1 The action

We first verify that both actions are defined in the same dimension and with the same matter content, and then we proceed to consider their form. Both actions are essentially 2d once symmetry is accounted for: in spherical collapse the two essential coordinates are the Lorentzian  $(\tilde{r}, \tilde{t})$ , while in the merger they are the Euclidean  $(r, z)$ . The matter content in  $d$  dimensional spherical collapse is  $d$  dimensional metric plus a scalar field  $\Phi$ , which is exactly the matter content of  $D = d + 1$  gravity in the merger system once a dimensional reduction over the time coordinate is performed (the Kaluza-Klein vector field vanishes due to time reversal symmetry).

The precise form of the action is identical as well, since the scalar obtained from dimensional reduction is minimally coupled, exactly as in (2.6). In order to exhibit the

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<sup>4</sup>However, it is not implied that the Cauchy horizon is met by a static asymptotic observer at finite time.

precise relation between the fields we proceed to perform this dimensional reduction in the standard way

$$ds_D^2 = e^{2A} dt^2 + \tilde{ds}_d^2 \quad (3.1)$$

the action is  $S = 1/(16\pi G_d) \int \sqrt{g_d} d^d x e^A \tilde{R}_d$  where  $G_d = G_D/L$  and after Weyl rescaling

$$ds_d^2 = e^{2A/(d-2)} \tilde{ds}_d^2 \quad (3.2)$$

one obtains

$$S = 1/(16\pi G_d) \int \sqrt{g_d} d^d x \left( R_d - \frac{d-1}{d-2} (\partial A)^2 \right) \quad (3.3)$$

finally one may rescale  $A$  to obtain a canonically normalized  $\Phi$ ,

$$\Phi = \sqrt{\frac{2(d-1)}{(d-2)}} A, \quad (3.4)$$

yielding the action for spherical collapse (2.6), up to the different signatures (and a signature related sign). Alternatively, in the spherical collapse we may consider  $\Phi$ , the dilaton, to arise from a dimensional reduction over an additional dimension, which we parametrize by  $\tilde{z}$  (and is analogous with  $t$  in (3.1)).

Moreover, the isometries of spherical collapse and of the merger are identical:  $SO(D-3)_\Omega \times U(1)_t \equiv SO(d-2)_\Omega \times U(1)_{\tilde{z}}$ .

More explicitly, it is standard to give the ansatz for spherical collapse as

$$\begin{aligned} ds_d^2 &= -\alpha(\tilde{r}, \tilde{t})^2 d\tilde{t}^2 + a(\tilde{r}, \tilde{t})^2 d\tilde{r}^2 + \tilde{r}^2 d\Omega_{d-2}^2 \\ \Phi &= \Phi(\tilde{r}, \tilde{t}). \end{aligned} \quad (3.5)$$

When compared with (2.1,3.1) we see that in the standard ansatz the gauge freedom is used to set

$$\begin{aligned} e^{C + \frac{1}{d-2} A} &\rightarrow \tilde{r} \\ ds_{\tilde{t}, \tilde{r}}^2 &\rightarrow -\alpha(\tilde{r}, \tilde{t})^2 d\tilde{t}^2 + a(\tilde{r}, \tilde{t})^2 d\tilde{r}^2. \end{aligned} \quad (3.6)$$

### 3.2 Boundary conditions

In order to fully define the solutions we must supply boundary conditions. For the Choptuik solution the b.c. are

- In the scaling direction ( $\tilde{\rho}$  - see figure 4 the evolution leads to the critical solution, as one evolves towards the singularity due to the solution's attractor nature. It was shown [16, 13] that the attractor mechanism could be replaced by periodic b.c.
- In the “tangential” direction (such as  $\tilde{r}$  for Choptuik) the b.c. are regularity on the  $r = 0$  axis and analyticity on the (past) horizon.

When the global properties of the Choptuik solutions were analyzed [13, 14] it was found that the standard solution in the “past” patch could be smoothly continued into the “outer” patch (see figure 4) for the definition of these patches) delineated by the past and future horizons. However, the continuation of the standard Choptuik solution is not analytic on the future horizon. This raises the possibility to define a different “tangential” b.c.: analyticity on both future and past horizons in the outer patch, or alternatively time reversal symmetry and horizon regularity. We term the solution obtained with these b.c. “time-symmetric Choptuik” or “TS-Choptuik” for short.

For the critical merger solution the b.c. are very similar

- In the scaling direction ( $\rho$ ) we expect an attractor at criticality, or equivalently self-similarity and periodicity.
- In the “tangential” direction,  $\chi$  or  $z$ , the boundary conditions are reflection symmetry  $z \rightarrow -z$  or alternatively  $\chi \rightarrow \pi - \chi$ , together with regularity on the horizon, namely that as  $\chi \rightarrow 0$  there is no conical deficit angle in the Euclidean geometry.

We see that the Choptuik and merger critical solutions have the same b.c. in the scaling direction, but different ones in the tangential direction: Choptuik has axis regularity while the merger has time reflection symmetry.

Thus, *the merger becomes the TS-Choptuik* after the following analytic continuation

$$\begin{aligned} z &\leftrightarrow i\tilde{t} \\ t &\leftrightarrow i\tilde{z} \\ r &\leftrightarrow \tilde{r} . \end{aligned} \tag{3.7}$$

While the second analytic continuation is trivial as the fields do not depend on this coordinate, the first is non-trivial as it involves an essential coordinate ( $z$  or  $\tilde{t}$ ) and it is crucial that the fields are even in that coordinate in order to retain reality after analytic continuation. For example for the scalar field  $\Phi(-\tilde{t}) = \Phi(\tilde{t})$  and hence  $\Phi$  is a function of  $\tilde{t}^2$  (namely, there exists some real analytic function  $\hat{\Phi}$  such that  $\Phi(\tilde{t}) = \hat{\Phi}(\tilde{t}^2)$ ) and analytic continuation sends  $\tilde{t}^2 \rightarrow -\tilde{t}^2$  keeping the function real (namely  $\hat{\Phi}(\tilde{t}^2) \rightarrow \hat{\Phi}(-\tilde{t}^2)$ ).

Comments:

1. In all cases there are two boundaries in the tangential direction: axis and horizon. So far we paid most of the attention to the axis, while the boundary conditions on the horizon were always “regularity”. Note however, that “regularity” of the horizon has two different meanings: in the Lorentzian case it means that one can pass smoothly to Kruskal-like coordinates, while in the Euclidean we have the “no deficit angle” boundary condition for the scalar field that plays the role that  $g_{tt}$  has in ordinary static geometries.
2. I find it likely, though not self-evident that solutions of “standard” spherical collapse respect a  $\tilde{r} \rightarrow -\tilde{r}$  symmetry. It is certainly obeyed by smooth spherically symmetric

scalar fields in a flat background, but the extension to curved space-time is not obvious to me. If this reflection symmetry does indeed exist then the Choptuik critical solution can be analytically continued via  $\tilde{r} \leftrightarrow i r$ ,  $\tilde{z} \leftrightarrow i t$ ,  $\tilde{t} \leftrightarrow z$  to a variant of the merger where the  $z$ -reflection b.c. is replaced by an axis regularity b.c. at  $r = 0$ .

### 3.3 An example

Here I give an explicit example for the correspondence (double analytic continuation) between a merger metric<sup>5</sup> and the corresponding TS-Choptuik metric.

Consider the metric for a cone over  $\mathbf{S}^2 \times \mathbf{S}^{D-3}$  (2.3). Identifying the coordinates  $\chi, t$  as in [3] and working with a Lorentzian metric we get the merger-type metric

$$ds^2 = d\rho^2 + \frac{\rho^2}{D-2} [d\chi^2 - \cos^2(\chi) dt^2 + (D-4) d\Omega_{\mathbf{S}^{D-3}}^2] , \quad (3.8)$$

where  $\chi$  was chosen in a slightly non-standard way to belong to the range  $-\pi \leq \chi \leq \pi$  such that the  $\mathbb{Z}_2$  symmetry reflection symmetry acts simply as  $\chi \rightarrow -\chi$  and  $t$  was identified such that the metric is independent of  $t$  and moreover  $g_{tt}$  vanishes at the boundaries of  $\chi$  (the horizon). Now we perform the double analytic continuation (3.7) appropriate for the time-symmetric case, where  $\chi$  plays the role of  $z$  (the coordinate with the reflection symmetry) and find

$$ds^2 = d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{D-2} [-d\tilde{t}^2 + \cosh^2(\tilde{t}) d\tilde{z}^2 + (D-4) d\Omega_{\mathbf{S}^{D-3}}^2] . \quad (3.9)$$

Finally, performing a dimensional reduction over  $\tilde{z}$  according to (3.3) with  $e^{2A} = \tilde{\rho}^2 \cosh^2(\tilde{t})/(d-1)$ ,  $d = D-1$ , and then normalizing  $\Phi$  according to (3.4) we get

$$ds^2 = \left( \frac{\tilde{\rho}^2 \cosh^2(\tilde{t})}{d-1} \right)^{1/(d-2)} \left[ d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{D-2} (-d\tilde{t}^2 + (D-4) d\Omega_{\mathbf{S}^{D-3}}^2) \right] \\ \Phi = \sqrt{\frac{2(d-1)}{d-2}} \left( \log(\tilde{\rho}) + \log(\cosh(\tilde{t})) - \frac{1}{2} \log(d-1) \right) . \quad (3.10)$$

We note that the metric in the  $\tilde{\rho}, \tilde{t}$  plane (the outer wedge in figure 4) is conformal to the Rindler metric, namely a wedge in 2d Minkowski space.

### 3.4 Cones and GHP oscillations

The action and boundary conditions are (continuously) scale invariant. Therefore it is natural to start by looking for continuously self-similar (CSS) solutions, also known as cones. The most general CSS ansatz is

$$ds_d^2 = e^{2B_\rho(\chi)} \left( d\rho + \rho \hat{A}(\chi) d\chi \right)^2 + \rho^2 e^{2B_\chi(\chi)} d\chi^2 + \rho^2 e^{2C(\chi)} d\Omega_{d-2}^2 \\ \Phi(\rho, \chi) = \kappa_\phi \rho + \Phi(\chi) \quad (3.11)$$

---

<sup>5</sup>[23] gives significant evidence that this is actually *the* critical merger solution, namely the attractor.

where all fields  $B_\rho, B_\chi, \hat{A}, C, \Phi$  depend only on  $\chi$ . Examples of cones include the double-cone (2.3) and the Roberts solution [11]

$$\begin{aligned} ds^2 &= -du dv + \tilde{r}^2(u, v) d\Omega^2 \\ \tilde{r}^2(u, v) &= [(1 - p^2) v^2 - 2 v u + u^2] \\ \Phi &= \frac{1}{2} \log \frac{(1 - p) v - u}{(1 + p) v - u} , \end{aligned} \quad (3.12)$$

where  $p = 1$  is a critical value.

Assuming that the double cone is indeed the critical merger solution [23], the exponents  $s$  (2.5) which appear at the linearized level can be interpreted as follows. Take (2.5) and perform two substitutions. First substitute  $\epsilon \rightarrow \delta p := p - p^*$  for the deviations from the double cone. Second, replace  $\rho \rightarrow \rho/\rho_0$  in order for the expression to be dimensionally correct, and  $\rho_0$  will be interpreted as a length scale characteristic of the smooth cone, for example,  $\rho_0^{-2}$  could be a measure of its maximal curvature. The result is

$$\delta p \sim (\rho/\rho_0)^s \sim \rho_0^{-s} . \quad (3.13)$$

Therefore

$$\rho_0 \sim \delta p^{-1/s} . \quad (3.14)$$

In the theory of critical collapse an analogous relation defines the critical exponents  $\gamma, \Delta$

$$\rho_0 \sim \delta p^{\gamma(1 \pm i 2\pi/\Delta)} , \quad (3.15)$$

where  $\rho_0^{-2}$  is a measure of the maximal curvature above or below criticality, and  $\Delta$  is the log-period of the GHP oscillations. Normally one writes only the real part of the exponent  $\rho_0 \sim \delta p^\gamma$  and (3.15) is a compact form which includes also the GHP oscillations. Moreover for critical collapse  $\Delta$  measured from GHP oscillations is the same as the log-period of the critical (DSS) solution,  $Z_*$ .

Comparing (3.13, 3.15) we find that  $s$  is a complex quantity which naturally combines the two scaling constants  $\gamma, \Delta$  through

$$-\frac{1}{s} = \gamma \left( 1 \pm i \frac{2\pi}{\Delta} \right) . \quad (3.16)$$

Therefore

$$\gamma = -\Re \left( \frac{1}{s} \right) \quad (3.17)$$

$$\frac{2\pi}{\Delta} = \frac{\Im(1/s)}{\Re(1/s)} . \quad (3.18)$$

Combining (3.17, 3.18) with the explicit expressions for  $s$  (2.5) we may predict off-critical oscillations for the merger at  $D < 10$ , with the following critical exponents

$$\gamma = \frac{1}{4} \quad (3.19)$$

$$\frac{2\pi}{\Delta} = \sqrt{\frac{10 - D}{D - 2}} , \quad (3.20)$$

while for  $D \geq 10$  there are no oscillations and the critical exponent  $\gamma$  becomes

$$\gamma = -\frac{1}{s_+} = \frac{1}{4} \left( 1 + \sqrt{\frac{D-10}{D-2}} \right), \quad (3.21)$$

where  $s_+$  was substituted in (3.17) since it is leading for large  $\rho$ .

Equation (3.17) may be compared with (2.11), the well-known connection in the theory of critical collapse.

#### 4. Consequences and indications

The correspondence outlined in the previous section suggests certain new directions for numerical work.

- **Predictions for off-critical oscillations and for the scaling constants of the merger for  $D < 10$ .**

Off-critical oscillations in the merger system are predicted to be analogous with GHP oscillations [13, 15] in critical collapse. The predicted values of the two scaling constants are given in (3.19,3.20).

*Suggested numerical experiment.* Measure the scaling constants  $\gamma, \Delta$  from off-critical merger solutions for various dimensions  $D < 10$ , improving on the pioneering work of [17, 18]. According to (3.15)  $\gamma$  may be defined through the maximal curvature as one goes off criticality, exactly as in Choptuik scaling, and  $\Delta$  is defined such that the period in  $\log \delta p$  is  $\Delta/\gamma$ .

- **Critical dimension  $D^* = 10$  for merger, and a prediction for  $\gamma$ .**

For  $D \geq D^* = 10$  GHP oscillations are predicted to cease to exist as a consequence of the property described around (2.5). The prediction for  $\gamma$  becomes (3.21).

*Suggested numerical experiment.* Seek the off-critical behavior for the merger in dimensions  $D \geq D^* = 10$ .

- **Possible similarity between Choptuik scaling and the merger.**

The merger and Choptuik systems were shown to have the same action after analytic continuation, but different boundary conditions. At the time when this paper was conceived, the data regarding the Choptuik scaling constants in various dimensions was scarce, and seemed to compare surprisingly well with the prediction for the merger (3.20): the predicted  $\Delta$  for merger is  $\Delta|_{D=5} = \Delta|_{D=7} = 4\pi/\sqrt{15} \simeq 3.24$  while the available  $\Delta$ 's for critical collapse were  $\Delta(4d) \simeq 3.45$ ,  $\Delta(6d) = 3.03$  [19] and  $d = D - 1$  (the available data at the time is summarized in table 2). That led to the suggestion that perhaps the change in boundary conditions would affect the scaling constants only weakly, making the Choptuik constants always close to (3.17,3.20,3.21). Moreover it suggested the possibility that critical dimensions which are known for the merger system [3, 20] will appear in Choptuik scaling as well.



	$\Delta$	$\gamma$
<b>4d</b>	3.45	0.374
<b>6d</b>	3.03	0.424

**Table 2:** Scaling constants for the Choptuik critical collapse in 4d [1] and 6d [19] which were available at the time this paper was conceived.  $\Delta$  is the log-period, and  $\gamma$  is the scaling exponent. By now a strikingly precise determination of  $\Delta(d = 4)$  is available:  $\Delta(d = 4) \simeq 3.445452402(3)$  [14]. In higher dimensions  $\gamma$  is defined such that  $M \simeq (p - p_*)^{\gamma(d-3)}$ , namely,  $(p - p_*)^\gamma$  has length dimension 1. For newer data in various dimensions see [21].

Inspired by these ideas and motivated by the apparent success of the estimates in 4d, 6d Sorkin and Oren set out to measure the scaling constants  $\gamma, \Delta$  for critical collapse in  $d \leq 11$  [21] (see [24, 25] for previous attempts). They succeeded and their interesting results indicate that (3.20) is *not* a good estimator for the Choptuik  $\Delta$  and the good agreement in certain dimensions which was described above should be considered a coincidence. The results for  $\gamma$  are not very close either. This is not surprising in view of the differences between the systems both in b.c. and in the DSS vs. CSS nature of the critical solution.

Regarding a critical dimension their results are less conclusive. No phase transition to CSS was observed up to  $11d$ , but there are some indications for extrema of the scaling constants (as a function of dimension) shortly above  $11d$ . Note that the large  $d$  simulations are impeded by a growingly singular behavior at the  $\tilde{r} = 0$  axis.

## Acknowledgements

The author is indebted to Nissan Itzhaki for important discussions and participation in the early part of this work. It is a pleasure to thank Evgeny Sorkin for discussions, and him and Yonatan Oren for sharing the results of [21] prior to publication. I also wish to thank members of the theoretical physics group at the Hebrew University for various discussions.

It is a great pleasure to thank my brother, Boaz Kol, for a discussion on self-organized criticality, a central topic of his PhD thesis.

BK is supported in part by The Israel Science Foundation (grant no 228/02) and by the Binational Science Foundation BSF-2002160.

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